INTRODUCTION

Use of auxiliary information at the estimation stage improves the efficiency of estimates of the population parameters of variable of interest. Conventional ratio, product, difference and linear regression estimators were introduced that utilize the information on auxiliary variable. The ratio estimator introduced by (Cochran, W.G. 1940) is more efficient if the study variable Y is positively correlated with auxiliary variable X, while the product estimator introduced by (Murthy, M. N. 1964) is more useful when the study variable Y is negatively correlated with auxiliary variable X. Linear regression estimator is useful when line of regression of Y on X is linear but does not pass through origin. Later on, various authors (Hartley, H. O., & Ross, A. 1954; Walsh, J.E. 1970; Ray, S.K. et al., 1979; Srivenkataramana, T., & Tracy, D.S. 1979; Vos, J. W. E. 1980; Ray, S. K., & Sahai, A. 1980; 9; Srivenkataramana, T. 1980; Singh, R. V. K., & Singh, B. K. 2007; Isaki, C. T. 1983; & Shan, D. N., & Gupta, M. R. 1987) introduced modified class of ratio and product type estimators. Such modified estimators are generally developed either using one or more unknown constants or introducing a convex linear combination of sample and population means of auxiliary characteristic with unknown weights. In both the cases, optimum choices of unknown constant were made by minimizing the mean square error of modified estimators so that they become more efficient than the conventional one. (Bahl, S., & Tuteja, R. 1991) Introduced exponential ratio-product type estimator and shown that both the estimators performed better than the conventional estimators. Later on, various authors (Upadhyaya, L. N. et al., 2011 & Solanki, R. S. et al., 2012) suggested exponential estimators in sampling theory. In this paper a combined exponential ratio-type estimator has been suggested for estimating finite population mean of characteristic under study.

Notation

Consider a finite population of size N which is divided into k strata such that $\sum_{i=1}^{k} N_h = N$

(h = 1, 2, 3, ..., k) let us select sample of size $n_h$ from $h^{th}$ stratum by SRSWOR such that $\sum_{i=1}^{k} n_h = n$. Let us further denote:

\[ \bar{y}_{st} = \sum_{i=1}^{k} W_h \bar{y}_h \], stratified sample mean of per unit

\[ \bar{y}_h = \frac{1}{n_h} \sum_{j=1}^{n_h} \bar{y}_{hj} \], sample mean of $h^{th}$ stratum

\[ \bar{P} = \sum_{h=1}^{k} W_h \bar{P}_h \], the population mean

Abstract: In this paper a combined exponential ratio-type estimator has been proposed for estimating population mean of characteristic under study. The expressions for bias and mean square error have been derived up to first order of approximation and found that the optimum mean square error of proposed estimator is equal to the mean square error of combined linear regression estimator. Efficiency of proposed estimator is compared theoretically with existing estimators and supported by numerical illustration.

Keywords: Simple random sampling, Ratio and regression-type estimator, Auxiliary information, mean squared error, Efficiency.
Where, \( \bar{y}_{h} = \frac{1}{N_{h}} \sum_{j=1}^{N_{h}} y_{hj} \) and \( W_{h} = \frac{N_{h}}{N} \)

\[ S_{y_{h}}^{2} = \frac{1}{N_{h}-1} \sum_{j=1}^{N_{h}} (\bar{y}_{hj} - \bar{y}_{h})^{2} ; \text{The population mean square of h}^{th} \text{stratum of characteristic under study.} \]

\[ S_{x_{h}}^{2} = \frac{1}{N_{h}-1} \sum_{j=1}^{N_{h}} (\bar{x}_{hj} - \bar{x}_{h})^{2} ; \text{The population mean square of h}^{th} \text{stratum of auxiliary variable.} \]

\[ S_{y_{h}x_{h}} = \sum_{j=1}^{N_{h}} (\bar{y}_{hj} - \bar{y}_{h})(\bar{x}_{hj} - \bar{x}_{h}) \]

\[ \theta_{h} = \frac{1}{n_{h}} - \frac{1}{N_{h}} \]

\[ R = \frac{\bar{y}}{\bar{x}} ; \text{The ratio of population means} \]

\[ C_{y_{h}} = \frac{S_{y_{h}}}{\bar{y}_{h}} ; \text{The coefficient of variation of h}^{th} \text{stratum of characteristic under study.} \]

\[ C_{x_{h}} = \frac{S_{x_{h}}}{\bar{x}_{h}} ; \text{The coefficient of variation of h}^{th} \text{stratum of auxiliary variable.} \]

\[ \rho = \frac{S_{y_{h}x_{h}}}{\bar{y}_{h}S_{x_{h}}} ; \text{The Correlation coefficient between the value of auxiliary variable and value of characteristic under study.} \]

Existing Estimators

(Bahl, S., & Tuteja, R. 1991) Suggested exponential ratio-type and product –type estimators for population mean \( \bar{Y} \), respectively, as

\[ \bar{y}_{re} = \bar{y} \exp \left( \frac{\bar{X} - \bar{Y}}{\bar{X} + \bar{Y}} \right) \quad \text{and} \quad \bar{y}_{pe} = \bar{y} \exp \left( \frac{\bar{X} - \bar{Y}}{\bar{X} + \bar{Y}} \right) \]

(1)

The mean square error of \( \bar{y}_{re} \) and \( \bar{y}_{pe} \) are given as:

\[ MSE(\bar{y}_{re}) = \left( \frac{1}{n} - \frac{1}{N} \right) \left( S_{v}^{2} + \frac{1}{4} R^{2} S_{x}^{2} + RS_{xy} \right) \]

(2)

\[ MSE(\bar{y}_{pe}) = \left( \frac{1}{n} - \frac{1}{N} \right) \left( S_{v}^{2} + \frac{1}{4} R^{2} S_{x}^{2} + RS_{xy} \right) \]

(3)

(Kadilar, G. Ö. 2016) Suggested an exponential estimator as

\[ \bar{y}_{pr} = \bar{y} \left( \frac{\bar{x}}{\bar{x} + \bar{y}} \right)^{a} \exp \left( \frac{\bar{x} - \bar{y}}{\bar{x} + \bar{y}} \right) \]

(4)

(Bahl, S., & Tuteja, R. 1991) Exponential ratio-type and product –type estimators for population mean \( \bar{Y} \) under stratified random sampling become:

\[ \bar{y}_{res} = \bar{y}_{st} \exp \left( \frac{\bar{x} - \bar{y}_{st}}{\bar{x} + \bar{y}_{st}} \right) \]

(6)

\[ \bar{y}_{pes} = \bar{y}_{st} \exp \left( \frac{\bar{x} - \bar{y}_{st}}{\bar{x} + \bar{y}_{st}} \right) \]

(7)

The mean square error of \( \bar{y}_{res} \) and \( \bar{y}_{pes} \) are given as:

\[ MSE(\bar{y}_{res}) = \sum_{h=1}^{k} W_{h} \left( \frac{1}{n_{h}} - \frac{1}{N_{h}} \right) \left( S_{v_{h}}^{2} + \frac{1}{4} R^{2} S_{x_{h}}^{2} - RS_{xyh} \right) \]

(8)

\[ MSE(\bar{y}_{pes}) = \sum_{h=1}^{k} W_{h} \left( \frac{1}{n_{h}} - \frac{1}{N_{h}} \right) \left( S_{v_{h}}^{2} + \frac{1}{4} R^{2} S_{x_{h}}^{2} + RS_{xyh} \right) \]

(9)
Proposed Estimator

Motivated from the work done by Gamze Ozel Kadilar (2016) and others, a combined exponential ratio-type estimator for population mean has been proposed as:

\[ t_p = \bar{y}_{st} \left( \frac{x_{st}}{x} \right)^\alpha \exp \left( \frac{x_{st} - \bar{x}}{x + \bar{x}_{st}} \right) \]  

(10)

Bias and Mean Square Error (Mse) of \( \bar{y}_{RP} \)

To obtain the bias and mean square error, let us suppose

\[ \bar{y}_{st} = \bar{y}(1 + e_0) \text{ and } \bar{x}_{st} = \bar{x}(1 + e_1) \]  

(11)

\[ E(e_0) = E(e_1) = 0 \]

\[ E(e_0^2) = \frac{1}{\bar{y}^2} E(\bar{y}_{st} - \bar{y})^2 = \frac{1}{\bar{y}^2} V(\bar{y}_{st}) \]  

(12)

Similarly \( E(e_1^2) = \frac{1}{\bar{x}^2} V(\bar{x}_{st}) \) & \( E(e_0 e_1) = \frac{1}{\bar{y}\bar{x}} \text{Cov}(\bar{y}_{st}\bar{x}_{st}) \)

\[ t_p = \bar{y}(1 + e_0) \left( \frac{\bar{x}_{(1+e_1)}}{x} \right)^\alpha \exp \left( \frac{\bar{x}_{st} - \bar{y}(1+e_0)}{x + \bar{x}_{st}(1+e_1)} \right) \]

\[ = \bar{y}(1 + e_0)(1 + e_1 + \alpha e_1^2 + \frac{\alpha(\alpha - 1)}{2} e_1^2 + \cdots)(1 - \frac{e_1}{2} + \frac{3}{8} e_1^2) \]

\[ = \bar{y}(1 + e_0 + e_1 + \alpha e_0 + \frac{\alpha(\alpha - 1)}{2} e_1^2)(1 - \frac{e_1}{2} + \frac{3}{8} e_1^2) \]

\[ = \bar{y}(1 + e_0 + e_1 + \alpha e_0 + \frac{\alpha(\alpha - 1)}{2} e_1^2 - \frac{e_1}{2} - \frac{e_1 e_0}{2} - \frac{\alpha}{2} e_1^2 - \frac{3}{8} e_1^2) \]

Now, \( (t_p - \bar{y}) = e_0 + e_1 + \alpha e_0 + \frac{\alpha(\alpha - 1)}{2} e_1^2 - \frac{e_1}{2} - \frac{e_1 e_0}{2} - \frac{\alpha}{2} e_1^2 - \frac{3}{8} e_1^2 \)

(13)

The bias of the proposed estimator \( t \) to terms of order \( n^{-1} \) can be obtained by taking the expectation of (7) and substituting results obtained in (5) as

\[ B(t_p - \bar{y}) = \frac{2(2a-1)}{\bar{y}^2} Cov(\bar{y}_{st}\bar{x}_{st}) \]

(14)

Now, squaring both sides of (7) and taking expectation and using results in (5), the MSE of the proposed estimator \( t \) to terms of order \( n^{-1} \) is obtained as

\[ \text{MSE}(t_p) = E(t_p - \bar{y})^2 \]

\[ = E(\bar{y}(1 + e_0 + \frac{(2a-1)}{2} e_1)^2) \]

\[ = E(\bar{y}(\bar{y}_0^2 + \frac{(2a-1)^2}{4} e_1^2 + (2a - 1) e_1 e_0)) \]

\[ = \bar{y}^2 \left( E(e_0^2) + \frac{(2a-1)^2}{4} E(e_1^2) + (2a - 1) E(e_1 e_0) \right) \]

\[ = \bar{y}^2 \left( 1 \frac{1}{\bar{y}^2} V(\bar{y}_{st}) + \frac{(2a-1)^2}{4} \frac{1}{\bar{x}^2} V(\bar{x}_{st}) + (2a - 1) \frac{1}{\bar{y}\bar{x}} \text{Cov}(\bar{y}_{st}\bar{x}_{st}) \right) \]

\[ = V(\bar{y}_{st}) + R^2 \frac{(2a-1)^2}{4} V(\bar{x}_{st}) + R(2a - 1) \text{Cov}(\bar{y}_{st}\bar{x}_{st}) \]

\[ = \sum_{h=1}^{k} W_h \left( \frac{1}{n_h} - \frac{1}{N_h} \right) \left( S_{yh}^2 + R^2 S_{xh}^2 + \frac{(2a-1)^2}{4} S_{xyh} + R(2a - 1)S_{xyh} \right) \]

(15)

Optimum value of \( \alpha \)

\[ \frac{\partial \text{MSE}(t_p)}{\partial \alpha} = 0 \]

Hence, \( R^2 \frac{4(2a-1)}{4} V(\bar{x}_{st}) + 2R \text{Cov}(\bar{y}_{st}\bar{x}_{st}) = 0 \)

\[ \alpha = \frac{R^2 V(\bar{x}_{st}) - 2R \text{Cov}(\bar{y}_{st}\bar{x}_{st})}{4} \]

\[ = \frac{1}{2} \frac{R^2 V(\bar{x}_{st})}{R \text{Cov}(\bar{y}_{st}\bar{x}_{st})} \]

\[ = \frac{1}{2} \frac{1}{R \beta} \]

(16)
Where, $A = \text{Cov}(\bar{y}_x, \bar{x}_x)$ and $B = V(\bar{x}_x)$

Minimum MSE of $t_p$ by substituting the optimum value of $\alpha$

We have $\text{MSE}(t_p)_{\text{min}} = V(\bar{y}_x) + R^2 \left( \frac{\text{Cov}(\bar{y}_x, \bar{x}_x)}{4} \right) B + R \left( \frac{1}{2} - \frac{1}{R} \right) A$

$$= V(\bar{y}_x)(1 - \rho^2)$$

$$= \sum_{h=1}^{k} \left( \frac{1}{n_h} - \frac{1}{N_h} \right) W_h^2 S^2_{yh}(1 - \rho^2)$$

Where, $\rho^* = \frac{\text{Cov}(\bar{y}_x, \bar{x}_x)}{V(\bar{x}_x) V(\bar{y}_x)}$

That is the MSE of combined linear regression estimator. Hence for the optimum value of $\alpha$ the proposed estimator $t$ is equally efficient as the combined linear regression estimator.

**Efficiency Comparison**

In this section efficiency of the proposed estimator is compared with that of existing estimators and conditions are obtained under which the proposed estimator is more efficient.

$$\text{MSE} (\bar{y}_{\text{Res}}) - \text{MSE} (t_p) > 0$$

$$\sum_{h=1}^{k} W_h^2 \left( \frac{1}{n_h} - \frac{1}{N_h} \right) \left( S^2_{yh} + \frac{1}{4} R^2 S^2_{xh} - R S_{xyh} \right) - \sum_{h=1}^{k} W_h^2 \left( \frac{1}{n_h} - \frac{1}{N_h} \right) \left( S^2_{yh} + \frac{2(\alpha - 1)^2}{4} + R(2\alpha - 1)S_{xyh} \right) > 0$$

On simplification, $\alpha = 1 + \frac{2 \sum_{h=1}^{k} W_h^2 \left( \frac{1}{n_h} - \frac{1}{N_h} \right) S_{xyh}}{\sum_{h=1}^{k} W_h^2 \left( \frac{1}{n_h} - \frac{1}{N_h} \right) S^2_{xh}}$

Again,

$$\text{MSE} (\bar{y}_{\text{Res}}) - \text{MSE} (t_p) > 0$$

$$\sum_{h=1}^{k} W_h^2 \left( \frac{1}{n_h} - \frac{1}{N_h} \right) \left( S^2_{yh} + \frac{1}{4} R^2 S^2_{xh} + R S_{xyh} \right) - \sum_{h=1}^{k} W_h^2 \left( \frac{1}{n_h} - \frac{1}{N_h} \right) \left( S^2_{yh} + \frac{2(\alpha - 1)^2}{4} + R(2\alpha - 1)S_{xyh} \right) > 0$$

Similarly on simplification, $\alpha = 1 - \frac{2 \sum_{h=1}^{k} W_h^2 \left( \frac{1}{n_h} - \frac{1}{N_h} \right) S_{xyh}}{\sum_{h=1}^{k} W_h^2 \left( \frac{1}{n_h} - \frac{1}{N_h} \right) S^2_{xh}}$

**Empirical Study**

The efficiency of the proposed estimator has been compared with existing estimators considered in this paper. The Descriptions of the population is given as

**Data Statistics**

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CONCLUSION

From above mentioned table it is found that the proposed estimator has greater relative efficiency than other existing estimator considered in this paper and also for optimum value of $\alpha$ the proposed estimator is equally efficient as combined linear regression estimator.

Thus the proposed estimator can be recommended for estimating the population mean of variable of interest in survey sampling.

REFERENCES